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Broken Symmetry of Lie Groups of Transformation Generating General Relativistic Theories of Gravitation[1]

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Intransitive Lie groups of transformations have invariant varieties which in suitable cases can be considered as space–times of a universe. The physical laws in the latter are expressed in terms of group theoretical notions. Theorems on the coincidences of group trajectories and geodesics are derived. The groups of linear transformations of the space of basis vectors are used as gauge groups to break the symmetry of the group of transformations and of their natural metric. It is shown that in case of the de Sitter group and its adjoint group as gauge group, one obtains in this way general relativistic theories of gravitation, especially Einstein's theory. More general aspects of the formalism are discussed.

I. INTRODUCTION AND SUMMARY

The first formulation of Einstein's general theory of relativity as a gauge theory of the Lorentz group has been suggested by Utiyama soon after Yang and Mills (1954) presented their generalization of Weyl's gauge method. Integral formulations of a gauge theory of gravitation were given by Lubkin (1963) and later by Yang (1974) with the general linear group, acting on the space of tangent vectors, as gauge group. The author later suggested the linear group acting on the spinors for a (quasi) unified gauge theory of gravitation and electromagnetism (Halpern, 1977b, 1979d).

Dirac's method to generalize matter field equations to invariance groups other than the Lorentz group, especially the de Sitter group and the conformal group (Dirac, 1935), inspired the author to create a de Sitter covariant formulation of general relativity (Halpern, 1977a). Preliminary

[1]Article written in memoriam of B. Jouvet of the Collège de France

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considerations along this line were already made by Lubkin (1971) without the author's knowledge. The author started out from a de Sitter-covariant version of Pauli's (1921) formulation of the principle of equivalence and generalized Dirac's construction to curved space (1977a). There are no doubt as many different formulations of this modified principle of equivalence possible as of the original one, but they are also largely equivalent.

An important motive for such group-covariant constructions, besides its obvious use in attempts of constructing unified field theories and its general interest in connection with the asymptotic properties of space–time (Lubkin, 1971) was, to create techniques that would allow to formulate a field theory based on Dirac's large-number hypothesis (Halpern, 1978b). The latter requires a theory that is largely in agreement with Einstein's theory locally but not globally.

The author gave then a general formulation of group-covariant field equations in which only notions pertaining to the group of transformation occur (Halpern, 1978a, b; 1979a–c); even the metric of homogeneous space–time is expressed in terms of quantities belonging to group theory and so is the universe itself. A short account of this formulation is given in Section 2. It forms the basis for the gauge method of the author which is again a gauge theory of the group of transformation whereby the symmetry of the group action on the space in broken (Halpern, 1978a, b; 1979a–c). This development is presented in Section 4. In Section 3 and Appendix A some necessary mathematical preliminaries are developed. We would like to stress that the breaking of the symmetry occurs in this theory only for the group of transformations—not for the abstract group and its group space which retain their symmetry for local action on the representation space.

The largest gauge group is the general linear group with the dimension of the invariance group.

The most desirable gauge group for physics is, however, not the widest but the narrowest which yields all physical results. The author has suggested the adjoint group of the group of transformations (Halpern, 1978a; 1979a, b). This case is treated in Section 5, and it is shown in Section 6 that for the case of the de Sitter group (and related groups) the equations can be solved in a special gauge and the resulting theory is either the general theory of relativity with a de Sitter background, or a related nonlinear theory.

Before its solution, the theory assumes the special form of a tetrad theory with a gauge potential. Such a gauge potential of tetrad fields has been considered earlier by the author in connection with problems in Møllers tetrad formulation of gravitational theory and the Lorentz group (Møller, 1969; Halpern and Miketinač, 1970). There is else no relation between the two cases. The method presented here gives, however, not only a new approach to general relativity and its nonlinear generalization. The group of transformations considered need not act on space–time alone; it can be a much wider group. The method of symmetry breaking provides a wider—if not a new—outlook of how gravitational fields are related to symmetry breaking and relative dislocations of the axes at different points of the higher-dimensional representation space. A wider outlook is also obtained for the unification of gravitation with electromagnetism and other fields and for the law of motion in its relation to trajectories of group generators; but hitherto these possibilities have not yet led to any significant improvements. They have therefore been limited here to brief remarks. The work on the present approach has only begun and a host of new possibilities have still to be investigated.

The notation follows closely that of Eisenhart (1933) with the exception that Latin letters are used for the space of basis vectors of the group.

2. MINIMAL INVARIANT VARIETIES AS MODELS OF THE SPACE–TIME OF THE UNIVERSE

Consider a continuous group of transformations G_r with r essential parameters, acting on a n-dimensional space V_n. The rank of the matrix of base vectors: (ξ_A^i) $(i=1\dots n, A=1\dots r)$ of G_r be $q<r<n$ so that the group is intransitive and there exist q-dimensional invariant varieties.

One can in general construct a metric of V_n such that G_r is a group of motion and each of the family of invariant varieties is a q-dimensional Riemannian subspace V_q imbedded in V_n. The case $q=4$ and signature $+2$ makes such a V_4 of suitable extensions a candidate for a model of space–time of a universe. One can then introduce a coordinate system in V_n such that everywhere

$$\xi_S^m=0, \qquad \frac{\partial \xi_R^i}{\partial x^m}=0, \qquad g_{mm}=\pm 1, \qquad g_{im}=0 \qquad (i\neq m)$$

$$(S=1\dots r, m=q+1,\dots,n, i=1\dots n, R=1\dots r) \tag{2.1}$$

because the generators $X_R=\xi_R^l\partial/\partial x^l$ of G_r act only within each invariant variety and because Killing's equations

$$\frac{\partial g_{ik}}{\partial x^l}\xi_S^l+g_{lk}\frac{\partial \xi_S^l}{\partial x^i}+g_{il}\frac{\partial \xi_S^l}{\partial x^k}=0 \tag{2.2}$$

are satisfied. We shall consider here only semisimple groups for which a nonsingular matrix

$$\gamma_{RS} = C^{U}_{RV} C^{V}_{SU} \tag{2.3}$$

can be formed out of the structure constants C^{S}_{RT}.

The author (Halpern, 1978a, b; 1979a–c) has expressed the contravariant metric tensor as

$$g^{ik} = C \xi^{i}_{R} \gamma^{RS} \xi^{k}_{S} \tag{2.4}$$

with

$$\gamma^{RS} \gamma_{ST} = \delta^{R}_{T} \tag{2.3a}$$

in the coordinates satisfying equations (2.1). C depends only on $x^m (m > q)$ and the ξ_R only on $x^i (i \leqslant q)$. For a given minimal invariant variety V_4, C is thus constant. The proof that (2.4) satisfies Killing's equations is furnished in Appendix A.

The metric of V_4 is thus expressed in terms of generators and structure constants of G_r. Field equations can now be constructed that consist only of group covariant expressions. Lie derivatives should replace all other derivatives.

The Lagrangian density of a scalar field becomes

$$\mathfrak{L} = g^{1/2} \gamma^{RS} \xi^{i}_{R} \left(\frac{\partial}{\partial x^{i}} \phi^{*} \right) \xi^{k}_{S} \left(\frac{\partial}{\partial x^{k}} \phi \right) \tag{2.5}$$

A Lie derivative of spinors on the V_4 of a G_r has also been constructed. We shall not discuss it here.[2]

Having achieved that task for the field equations, we would like to express the equation of motion of a classical body as well in a G_r-covariant way. The timelike geodesics of V_4, however, do not in general coincide with group trajectories. Agreement with the well-established results of general relativity can only be achieved if trajectories exist that approximate these geodesics well enough. We consider in the following only the case where every geodesic of V_4 is a group trajectory. This situation is further explored in the next section.

[2]See Halpern (1978a, b, c; 1979a–c). A Lie derivative of spinors which is apparently related to the authors has also been suggested by W. Unruh (private communication).

3. GEODESICS AND GROUP TRAJECTORIES

The basis vectors ξ_R of G_r are determined only up to a nonsingular r-dimensional linear transformation C. The theory of Lie groups of transformations is formulated covariantly with respect to such linear transformations. The transformation C can be chosen so that at a given ordinary point P_0 of V_4, $q=4$ of the basis vectors, let us say ξ_B $(B=1,\ldots,4)$, are orthonormal and the remaining ξ_M $(M=q+1,\ldots,r)$ vanish there. We denote such a system of basis vectors as "special" at P_0. A special system is of course not uniquely determined; apart from rotations, vectors ξ_M $(M>q)$ can arbitrarily be added to the ξ_B.

> *Theorem.* The group trajectory of a base vector ξ is a geodesic through a point P_0, if a special system of base vectors exist there such that

$$[\xi_B,\xi]^i g_{ik}\xi^k=\left(\xi_B^m\frac{\partial}{\partial x^m}\xi^i-\xi^m\frac{\partial}{\partial x^m}\xi_B^i\right)g_{ik}\xi^k=0 \quad (B=1\cdots q=4) \tag{3.1}$$

Proof. (1) The condition of equation (3.1) is satisfied at every point of the trajectory if it is satisfied at P_0, because none of its points is preferred. Indeed at any point P_1 of the trajectory the linear transformation of the base vectors by the adjoint group of G_r with the group element which brings P_1 into P_0 along the trajectory produces such a system at P_1. We shall show this for an infinitesimal displacement along the trajectory. We can always choose ξ itself as one of the vectors (let us say $\xi_1^i=\delta_1^i$) of the special system pointing in the 1-direction. At a neighboring point P_1 of the path,

$$x^i=x_0^i+\delta_1^i\delta\tau \tag{3.2}$$

the components of all vectors differ by

$$\delta\xi_R^i=\frac{\partial\xi_R^i}{\partial x^1}\delta\tau \qquad (R=1\ldots r) \tag{3.2a}$$

the transformation of the adjoint group alters them by

$$\delta\xi_R^i=C_{R1}^S\xi_S^i\delta\tau=\left(\frac{\partial\xi_1^i}{\partial x^k}\xi_R^k-\frac{\partial\xi_R^i}{\partial x^1}\right)\delta\tau \tag{3.2b}$$

clearly a coordinate transformation with

$$\frac{\partial x'^i}{\partial x^k} = -\frac{\partial \xi_1^i}{\partial x^k} \tag{3.2c}$$

restores then all the values of the transformed components to their values at P_0. The transformation of the adjoint group leaves the structure constants unchanged because of Jacobi's identities so that even the metric (2.4) is the same as at P_0 after the transformations and (3.1) is satisfied at P_1.

(2) To prove that the trajectory is a geodesic if and only if equation (3.1) is fulfilled at every one of its points we remember that every ξ in V_4 is the symbol of the group of motion and contract Killing's equation (2.2) with ξ^i and ξ^k to obtain

$$\xi^i \xi^k \frac{\partial g_{ik}}{\partial x^m} \xi_R^m + 2 g_{mk} \frac{\partial}{\partial x^i}(\xi_R^m) \xi^i = 0 \tag{2.2a}$$

with the help of (3.1) we obtain from this

$$\frac{\partial}{\partial x^m}(\xi^i g_{ik} \xi^k) \xi_R^m = 0 \qquad \text{implying } \frac{\partial}{\partial x^j}(\xi^i g_{ik} \xi^k) = 0 \qquad (j = 1 \ldots 4) \tag{3.3}$$

One can thus choose the parameter such that

$$\dot{x}^k = \xi^k$$

$$\ddot{x}^k = \frac{\partial \xi^k}{\partial x^m} \xi^m$$

and

$$\Gamma^k_{im} \dot{x}^i \dot{x}^m = \frac{1}{2} g^{kj} \left(2 \frac{\partial g_{ij}}{\partial x^m} - \frac{\partial g_{im}}{\partial x^j} \right) \xi^i \xi^m \tag{3.3a}$$

Because of (3.3):

$$-\frac{\partial g_{im}}{\partial x^k} \xi^i \xi^m = 2 g_{im} \xi^i \frac{\partial \xi^m}{\partial x^j} \tag{3.3b}$$

and because of (2.2):

$$\frac{\partial g_{ij}}{\partial x^k} \xi^i \xi^k = -\left(\xi^i g_{kj} \frac{\partial \xi^k}{\partial x^i} + \frac{\partial \xi^k}{\partial x^j} g_{ik} \xi^i \right) \tag{2.2b}$$

together (3.3a), (3.3b), and (2.2b) show that the equation of the geodesic is satisfied:

$$g_{jk}(\ddot{x}^k + \Gamma^k_{im}\dot{x}^i\dot{x}^m) = 0 \tag{3.4}$$

We obtain immediately the following corollary.

Every geodesic through P_0 is a group trajectory if and only if a special system of base vectors exist so that (3.1) is fulfilled for any two vectors $\xi = \xi_A, \xi_B$ $(A, B = 1\ldots 4)$. Each vector of the subspace spanned by $\xi_1 \cdots \xi_4$ has a geodesic as trajectory. We consider here only groups that fulfill this condition. Is the possibility of motion of macroscopic bodies along non-geodesic timelike group trajectories in conflict with experience? The author has repeatedly pointed out that this need not necessarily be so (Halpern, 1978b; 1979b, c).

Consider a special system of base vectors at P_0 in coordinates that result in a Minkowski metric there. The geodesic motion of a macroscopic body can be the trajectory of a vector $C^R\xi_R$ with $C^R = 0$ for $R > 4, \gamma_{RS}C^RC^S = 1$. One may attribute to that motion a volume in phase space proportional to

$$\sum_{A=1}^{3} \frac{(C^A)^2}{\gamma_{RS}C^RC^S} \tag{3.5}$$

Consider now the same relations in case of a nongeodesic trajectory of the same initial velocity, where $C^R \neq 0$ also for $R > 4$. The phase space volume according to equation (3.5) will shrink relative to the geodesic case, in the limit the more the smaller the ratio of the radii of curvature of the trajectory and the universe (geodesic motion) is. We can only observe a radius of curvature far smaller than that of the universe, which would correspond to such a small relative volume in phase space that we have practically no chance to encounter it among a limited number of samples —just as we do not encounter a macroscopic quantity of gas in vacuum, that will contract. The example given here is a possible generalized law of motion which is in the spirit of the group theoretical approach; it can be studied best in case of the de Sitter group treated in later sections. The equations of the generalized free motion are nonlinear and of higher order. They are briefly stated for the de Sitter group.

4. BREAKING OF THE GROUP SYMMETRY BY THE GAUGE FORMALISM

We have been able to express the metric of space–time in terms of quantities belonging to the group of transformations itself; it is therefore suggestive to describe the breaking of the symmetry of space due to local inhomogenous matter distributions in terms of a symmetry breaking of the group of transformation which acts on space. (Not of the abstract group which is to remain intact to allow action on localized quantities. Even the group of transformation should still function locally on path segments).

The author has suggested using the covariance of the group theory with respect to linear transformations of the space of base vectors (see the first part of Section 3) to establish a formalism of the gauge type, which can describe the symmetry breaking of space and (in a still rather artificial way) relate it to the presence of matter (Halpern, 1978a, b; 1979a–c).

We start by performing at every point of V_4 independent linear transformations of the space of base vectors. The transformations affect thus all indices with capital Latin letters. We are able to uphold our formalism in spite of this manoeuver if we only replace derivatives of quantities with block indices by invariant derivatives. Suppose ξ_u transforms:

$$\xi_U(x)\to\xi'_U(x)=S_U^V[U^x(x)]\xi_V(x) \tag{4.1}$$

by some subgroup Γ of $GL(r)$ with canonical parameters u^α.[3] The invariant derivative is defined in a well-known way (DeWitt, 1963) with a potential $A_k^\alpha(x)$ and the generators G_α of Γ:

$$\xi_{U\cdot k}=\frac{\partial}{\partial x^k}\xi_U+A_k^\alpha(G_\alpha)_U^V\xi_V \tag{4.1a}$$

A transformation S of Γ transforms the potentials inhomogenously with respect to the adjoint group of Γ. Infinitesimally

$$\delta A_k^\alpha(x)=C_{\beta\gamma}^\alpha A_k^\beta\delta u^\gamma(x)-\frac{\partial}{\partial x^k}\delta u^\gamma(x) \tag{4.1b}$$

so that $\xi_{U\cdot k}$ transforms in the same way as ξ_U:

$$\xi_{U\cdot k}\to S_U^V\xi_{V\cdot k} \tag{4.1c}$$

The condition that the potential can be transformed to zero at all points x simultaneously by a transformation $S_U^V(x)$ is

$$F_{ik}^\alpha(x)=\frac{\partial}{\partial x^i}A_k^\alpha-\frac{\partial}{\partial x^u}A_i^\alpha+C_{\beta\gamma}^\alpha A_i^\beta A_k^\gamma=0 \tag{4.1d}$$

[3]To distinguish Γ from G_r we use Greek indices instead of Capital Latin indices

A transformation $S_U^V[u^\alpha(x)]$ in our unperturbed V_4 of G_r produces potentials which satisfy equation (4.1d). The commutation relations are now of the form:

$$\xi_R^m \xi_{U\cdot m}^i - \xi_U^m \xi_{R\cdot m}^i = C_{RU}^{\prime T} \xi_T^i \tag{4.2}$$

where $\xi, \xi_{\cdot k}$ are defined in equations (4.1), (4.1a), and (4.1c) and

$$C_{RU}^{\prime T} = S_R^Y S_u^Z (S^{-1})_V^T C_{YZ}^V \tag{4.2a}$$

depends on x but

$$C_{RU\cdot k}^{\prime T} \equiv 0 \tag{4.2b}$$

Killing's equations assume the form

$$\frac{\partial g_{ik}}{\partial x^m} \xi_S^m + g_{mk} \xi_{S\cdot i}^m + g_{im} \xi_{S\cdot k}^m = 0 \tag{4.2c}$$

To break the symmetry of V_4 we abandon equations (4.1) and (4.1d) keeping, however, equations (4.1a)–(4.1c), (4.2), (4.2a)–(4.2c) for a potential which has now to be determined from field equations. Also the ξ_R and the $S_U^V(x)$ have to be obtained from field equations and from equations (4.2) and (4.2a). The generalized Killing equations (4.2c) imply furthermore that the metric is of the same form as equation (2.4) yet formed with the γ'^{RS} out of the primed structure "constants."

To obtain consistent field equations we have to form invariants out of the A_k^α, the ξ_R^i, and the S_U^V and add equations (4.2) with a Lagrangian multiplier, so that we can vary independently with respect to all the unknowns. We may in simpler cases solve equations (4.2), eliminate thereby some of the unknowns, and avoid the multipliers. The metric can always be constructed from the solutions.

The following two sections will provide examples for the procedures.

5. THE ADJOINT GROUP AS GAUGE GROUP

Every group G_r has a "natural" group of linear transformations acting on the space of its base vectors: The adjoint group of G_r has the same structure constants as G_r and thus an isomorphic law of composition of the group parameters (Eisenhart, 1933). We have seen moreover in Section 3 that the structure constants of G_r are not altered by the transformations of its adjoint group. The $S_U^V(x)$ which were only auxiliary variables, serving to exhibit the full invariance properties of the theory, are constants here.

The adjoint group is not transitive; this leads us to an additional restriction: We want to avoid solutions for the ξ_R^i which the gauge group even for one single point fails to transform to a set of base vectors of the unperturbed V_4. This means we must be able to transform the base vectors at every point into a special system (see Section 3) in which for $A=1,\ldots,q=4$ they form a Vierbein whereas those with $M>4$ vanish. This is because as we saw in Section 3, we can also always introduce a special system at one given point by a constant linear transformation and then transform this property to any other point of V_4 by a transformation of the adjoint group. The constant linear transformation fixes the components of the structure constants.

Thanks to our requirement we can introduce even in the general case a gauge for which at every point the first four base vectors form a Vierbein:

$$\xi_A^i = h_A^i \ (A,B=1\ldots4), \qquad g_{ik}\xi_A^i\xi_B^k = \eta_{AB}$$
$$\xi_M^i = 0 \qquad (M>4) \tag{5.1}$$

Written explicitly equations (4.2) in such a gauge are

$$h_A^m\left(\frac{\partial}{\partial x^m}h_B^i + A_m^E C_{BE}^D h_D^i\right) - h_B^m\left(\frac{\partial}{\partial x^m}h_A^i + A_m^E C_{AE}^D h_D^i\right) = C_{AB}^D h_D^i \tag{5.2a}$$

$$h_A^m\left(\frac{\partial}{\partial x^m}\xi_M^i + A_m^E C_{ME}^D h_D^i\right) = C_{AM}^D h_D^i \tag{5.2b}$$

$$(A,B,D,E,=1\ldots4,\ M=5\ldots r)$$

The Greek indices in equation (4.1) are replaced for the adjoint group by Latin indices because structure constants are the same as for G_r.

The equations in this form are much simplified. They will be solved for the de Sitter group in the next section.

6. THE de SITTER GROUPS AS AN EXAMPLE

The de Sitter and anti-de Sitter groups are five-dimensional orthogonal groups with signatures +3 and +1, respectively. The basis vectors are most simply expressed in five-dimensional Cartesian form, labeled by double indices which we denote here by two Greek letters:

$$\xi_{[\alpha,\beta]}^i = X^m\left(\eta_{m\beta}\delta_\alpha^i - \eta_{m\alpha}\delta_\beta^i\right) \qquad (i,m,\alpha,\beta=1\ldots5) \tag{6.2}$$

If we use the labels only as symbols or in summation we shall write, however, Latin letters as before. The structure constants are

$$C^{[\epsilon,\phi]}_{[\alpha,\beta][\gamma\delta]} = \tfrac{1}{2}\left[\eta_{\alpha\gamma}\left(\delta^{\epsilon}_{\beta}\delta^{\phi}_{\beta} - \delta^{\epsilon}_{\delta}\delta^{\phi}_{\beta}\right) - (\gamma\leftrightarrow\delta)\right] - \tfrac{1}{2}\left[(\alpha\leftrightarrow\beta)\right] \qquad (6.2a)$$

The geodesic condition equation (3.1) is fulfilled here.

$$\gamma_{[\alpha,\beta][\delta,\epsilon]} = 4(\eta_{\alpha\epsilon}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\epsilon}) \qquad (6.2b)$$

double index summations are performed over each of the two letters.

The minimal invariant varieties are generalized spheres:

$$\eta_{ik}X^iX^k = \pm R^2 \qquad (i,k=1\ldots n=5) \qquad (6.2c)$$

(upper sign de Sitter, lower, anti-de Sitter space), on which one can use conformal coordinates by introducing

$$x^k = \frac{2X^k}{R+X^5}, \qquad x^5 = R \qquad (k=1\ldots 4) \qquad (6.2d)$$

and choosing $R = R_0 = \text{const}$. The metric on the invariant variety is then

$$g^{ik} = \eta^{ik}\frac{(1\pm\sigma^2/4)^2}{R^2}, \qquad \sigma^2 = \eta_{lm}x^lx^m \qquad (i,k,l,m=1\ldots 4) \quad (6.2e)$$

and the base vectors are

$$\xi^i_{[\alpha,\beta]} = x^k(\eta_{k\beta}\delta^i_\alpha - \eta_{k\alpha}\delta^i_\beta) \qquad (\alpha,\beta=1\ldots 4) \qquad (6.2f)$$

and

$$\xi^i_{[\alpha,5]} = \pm\delta^i_\alpha\left(1\pm\frac{\sigma^2}{4}\right) + \frac{1}{2}x^ix^k\eta_{k\alpha} \qquad (6.2g)$$

We consider now the generalization discussed in Section 5 with the adjoint group as gauge group. Transforming at every point to a special system we give the four nonvanishing base vectors the indices $[\alpha,5]$ whereas all $\xi_{[\alpha,\beta]} = 0$ $(\alpha,\beta=1\ldots 4)$ (Remember the structure constants do not alter their form and values (6.2a) by the transformation, but the base vectors everywhere assume the components as at $x^i=0$ of equations (6.2f), (6.2g) expressed in general coordinates. We are able to solve equations (5.2a), (5.2b) for the potentials and find

$$A_m^{[\alpha,5]} = -h_{Am}\eta^{A\alpha} \qquad (A,\alpha = 1\ldots4) \tag{6.3a}$$

$$A_E^{[\alpha,\beta]} = A_k^{[\alpha\beta]}h_E^k = \gamma_{ABE}\eta^{A\alpha}\eta^{B\beta} \tag{6.3b}$$

with

$$\gamma_{ABE} = h_{A;k}^i h_{Bi} h_E^k \tag{6.4}$$

the coefficients of rotation (Eisenhart, 1964). All the 40 components of the potential are thus expressed in terms of the tetrads and their derivatives. We express now also the fields $F_{ik}^{[\alpha,\beta]}$ of equation (4.1d) by the tetrads:

$$F_{ik}^{[\alpha,5]} = 0 \tag{6.3c}$$

and

$$\begin{aligned} F_{ik}^{[\alpha,\beta]} &= \eta^{\alpha A}\eta^{\beta B}\left[\frac{\partial}{\partial x^i}\left(\gamma_{ABE}h_k^E\right) - \frac{\partial}{\partial x^k}\left(\gamma_{ABE}h_i^E\right) + \eta^{EF}(\gamma_{EAI}\gamma_{FBK} - \gamma_{EBI}\gamma_{FAK})h_i^I h_k^K\right] \\ &\quad + C_{[I,5]}^{[\alpha,\beta]}{}_{[K,5]}h_i^I h_k^K \\ &= \left(R_{pqki} - \mathring{R}_{pqki}\right)h^{\alpha p}h^{\beta q} \end{aligned} \tag{6.3d}$$

(all Latin and Greek indices $(1\ldots4)$ with R_{pqki} the Riemann tensor of the metric related to the tetrads and

$$\mathring{R}_{pqki} = \frac{1}{R_0^2}\left(g_{pk}g_{qi} - g_{pi}g_{qk}\right) \tag{6.3e}$$

In case of the unperturbed space V_4 $R_{pqki} = \mathring{R}_{pqki}$ and all F_{ik} vanish.

We consider the following invariants formed out of F_{ik} for a Lagrangian:

$$F_{ik}^R\gamma_{RS}F^{Sik} = R_{pqki}R^{pqki} + \frac{4}{R_0^2}R + \frac{24}{R_0^4} \tag{6.5}$$

which is a well-known quadratic Lagrangian with an admixture of an Einstein term (Halpern, 1977a, b) (necessary to avoid singularities in the solutions) and a cosmological constant.

Besides the Lagrangian of equation (6.5), which is of the Maxwell type, as it occurs in all gauge theories, there exists here another Lagrangian linear in the fields F_{ik}^{M}

$$F_{ik}^{[\alpha,\beta]}C_{[\gamma,5][\alpha,\beta]}^{[\varepsilon,5]}h^{\gamma i}h_{\varepsilon}^{k} \tag{6.5a}$$

which is equivalent to the Einstein Lagrangian with a cosmologic member. One obtains thus essentially Einstein's theory if one adds the conventional matter Lagrangians, e.g., equation (2.5) with a constant and varies the total Lagrangian with respect to the tetrad fields $h_{A(x)}^{i}$ and the matter fields.

The law of motion in this theoretical framework has, no doubt, a deeper basis than just the choice of a Lagrangian. We have given the condition for which in unperturbed space every geodesic is a group trajectory; the condition (3.1) is obviously fulfilled for the de Sitter group. The author has shown (Halpern, 1977a) that a principle of equivalence exists even for the de Sitter background, which means in analogy to Pauli's (1921) definition: Along the points of a geodesic an arbitrary metric can always be transformed into the metric of de Sitter space such that all its first derivatives vanish there.

This theorem contemplated from the point of view of the present paper leads to the following features:

(1) The potentials A_i^R can be transformed away on all points of a given line segment because the system of ordinary differential equations for the parameters of the adjoint group that achieves this has in general solutions.

(2) Killing's equations and the commutation relations of the $\xi_R^i(x)$ are then formally fulfilled along the points of the line and our considerations of Section 3 apply. A geodesic in the general case is then always a trajectory of a linear superposition of the transformed $\xi_R^i(x)$.

The system of differential equations that has as solutions all trajectories of the group is very complicated even in the unperturbed V_4. It is nonlinear of the fifth order in the de Sitter case. One obtains them by writing down the path of a general generator:

$$\dot{x}^i = \xi_{(x)}^i \equiv C^R \xi_R^i(x) \tag{6.6}$$

Assume a special system of generators at an initial point P_0.

> *Theorem.* A special system of generators can be introduced at any other point p of the trajectory such that the generator of the trajectory expressed in the transformed system is of the same form (6.6) as at P_0.

The proof follows the considerations and the theorem of Section 3: For an infinitesimal displacement along the trajectory the parameters of the adjoint group which achieve the transformation, $\delta\sigma^R$ are proportional to the C^R and thus

$$\delta C^R = C^R_{ST} C^S \delta\sigma^T = 0 \tag{6.7}$$

One can thus express the C^R at every point in terms of $\dot{x}$, $\ddot{x}$, and higher derivatives up to the fifth from the equations of the trajectory:

$$\xi^i_{;k}\xi^k = C^B_{AR} h^i_B C^A C^R \tag{6.8}$$

$$(A, B, D = 1 \ldots 4;\ R, S = 1 \ldots r)$$

$$(\xi^i_{;k}\xi^k)_{;m}\xi^m = C^D_{AR} C^B_{DS} C^A C^R C^S \tag{6.8a}$$

and its higher covariant derivatives in the ξ direction, the right-hand side of which consists of a chain of structure constants of the same form with one more member for each derivative. More details of these equations and their possible relation to physics are postponed to a subsequent work. The equations of motion of higher order pertain no doubt—if they really play a role in physics—to the nonlinear Lagrangian (6.5).

We finally mention a generalization of the above theory in which the gauge group consists of the direct product of the adjoint group and the group of scale transformations. The solution of the generalized commutation relations for the potentials A^R_k in terms of the tetrad fields and a simple vector gauge field can even be performed in such a case but the analogous results are not gauge covariant. We give here no details because in our opinion this approach will not result in a truly unified theory of gravitation and electromagnetism. Wider gauge groups either result in additional constraints between the potentials or they introduce torsion. The methods introduced here can be applied to a multitude of situations, the physical content of which should be investigated.

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APPENDIX A

Theorem. The Lie derivative of $\xi_R^i \gamma^{RS} \xi_S^k$ with respect to any basis vector ξ_U of G_r vanishes. (See Section 2.) [The result is well known for $r = n = q$ in group space (Eisenhart, 1933).]

Proof. The Lie derivative is $(C_{PU}^S \gamma^{PR} + C_{PU}^R \gamma^{SP}) \xi_S^i \xi_R^k$ and the expression in parenthesis equals $\gamma_{PQ,U} \gamma^{PR} \gamma^{QS}$ with

$$\gamma_{PQ,U} = C_{PU}^T \gamma_{TQ} + C_{QU}^T \gamma_{TP} = C_{TW}^V \left(C_{PU}^T C_{QV}^W + C_{QU}^T C_{PV}^W \right)$$

because of Jacobi's identifies this equals

$$\gamma_{PQ,U} = C_{TU}^V \left(C_{PW}^T C_{QV}^W + C_{QW}^T C_{PV}^W \right) - C_{WU}^T \left(C_{PT}^V C_{QV}^W + C_{QT}^V C_{PV}^W \right) = 0$$

because exchange of dummy indices results in terms that cancel.

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